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The mixed black hole partition function for the STU model

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Abstract: We evaluate the mixed partition function for dyonic BPS black holes using the recently proposed degeneracy formula for the STU model. The result factorizes into the OSV mixed partition function times a proportionality factor. The latter is in agreement with the measure factor that was recently conjectured for a class of $N = 2$ black holes that contains the STU model.

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Contents

1. Introduction

Some time ago it was conjectured that the partition function of four-dimensional BPS black holes with $N = 2$ supersymmetry, defined by

$$Z(p,\phi) = \sum_{\{q\}} d(p,q)\, \mathrm{e}^{\pi\, q_I \phi^I}\,, \tag{1.1}$$

is related to the topological string partition function [1]. The 'mixed' partition function (1.1) is based on an ensemble where the magnetic charges p and the electrostatic potentials ϕ are kept fixed. With respect to the magnetic charges one is therefore dealing with a micro-canonical ensemble, while the electric charges q are replaced by the continuous potentials ϕ. The $d(p,q)$ denote the microscopic black hole degeneracies for given magnetic and electric charges, p^I and q_I, respectively.

The logarithm of the mixed partition function can be viewed as a free energy function $\mathcal{F}_{\mathrm{E}}(p,\phi)$,

$$Z(p,\phi) \sim \mathrm{e}^{\pi\, \mathcal{F}_{\mathrm{E}}(p,\phi)}\,, \tag{1.2}$$

which can be identified with the one that exists in the context of the field-theoretic description of BPS black holes. The latter has a relation with the partition function $Z_{\mathrm{top}}(p,\phi)$ of the topological string [2], which indicates that the mixed partition sum (1.1) and the topological string are related. In [1], this relation was argued to take the following form,

$$\mathrm{e}^{\pi\, \mathcal{F}_{\mathrm{E}}(p,\phi)} = |Z_{\mathrm{top}}(p,\phi)|^2\,. \tag{1.3}$$

In subsequent developments it was realized that, while $Z(p,\phi)$ is invariant under certain imaginary shifts of the ϕ owing to the quantized nature of the electric charges, this invariance is in general not reflected in the free energy, so that one may have to include an explicit sum over these shifts on the right-hand side of (1.2). Furthermore it turns out that (1.3) cannot be an exact relation, but must involve a proportionality factor that plays

the role of a measure in the inverse Laplace transform that expresses the black hole degeneracies in terms of the free energy.[1] In [4] it was shown how to determine this measure from arguments based on duality in the context of the semiclassical approximation of the inverse Laplace transform. Independently, a direct evaluation of the mixed partition function from specific microscopic degeneracy formulae for dyonic black holes in $N = 4$ supersymmetric CHL models [5] (carried out in the context of an $N = 2$ formalism) revealed the presence of a measure factor [6, 4], which for large charges was in agreement with the prediction of [4] (see also [7]).

These matters warrant further study in the context of $N = 2$ black holes, where not many degeneracy formulae are known. A proposal for such a formula in the STU model [8, 9], which exhibits both exact S- and T-dualities, has been presented in [10]. This proposal was considered in a recent paper [3], where a number of subtleties were noted (to which we will turn in section 4), which, however, stayed short of evaluating the measure factor. It is the purpose of the present note to address this issue in more detail.

It is convenient to first discuss some common features shared by the degeneracy formulae for the $N = 4$ supersymmetric models [11 – 14] and the STU model.[2] They all involve an integral over appropriate 3-cycles of the inverse of an $\mathrm{Sp}(2,\mathbb{Z})$ automorphic form $\Phi_k(\rho,\sigma,\upsilon)$ of weight k (with suitable normalization),

$$d_k(K,L,M) = I_k(K,L,M) = \oint \mathrm{d}\rho\,\mathrm{d}\sigma\,\mathrm{d}\upsilon\,\frac{\mathrm{e}^{\mathrm{i}\pi[\rho\,K+\sigma\,L+(2\upsilon-1)\,M]}}{\Phi_k(\rho,\sigma,\upsilon)}\,, \tag{1.4}$$

where ρ,σ,υ are the complex elements of the period matrix of a genus-2 Riemann surface. The weight k and the number n of $N = 2$ physical vector supermultiplets are related by

$$n = 2(k+2) - 1\,. \tag{1.5}$$

For the STU model we have $k = 0$ and $n = 3$. For the $N = 4$ supersymmetric models this relation changes into $n = 2(k+2)+3$ to account for the four gauge fields associated with the extra gravitini supermultiplets; CHL black holes have been considered for $k = 1, 2, 4, 6, 10$. Upon including the $N = 2$ graviphoton, there are thus $n+1$ gauge fields, each associated with a magnetic and an electric charge. These charges are denoted by p^I and q_I, respectively, where $I = 0, 1, \ldots, n$.

The quantities K, L, M take discrete values proportional to the charge bilinears that transform as a triplet under an $\mathrm{SL}(2,\mathbb{Z})$ factor of the duality group (or a subgroup thereof) and are invariant under the remaining dualities. The explicit transformations of K, L, M are

$$\begin{aligned} K &\to d^2\,K + c^2 L + 2\,cd\,M\,,\\ L &\to a^2\,L + b^2\,K + 2\,ab\,M\,,\\ M &\to ac\,L + bd\,K + (ad+bc)M\,. \end{aligned} \tag{1.6}$$

For $\mathrm{SL}(2,\mathbb{Z})$ the parameters a, b, c, d are integer-valued parameters which satisfy $ad - bc = 1$. For the $N = 4$ models, these transformations constitute the S-duality group, which is

[1] There exist arguments of a more conceptual nature indicating that a modification of (1.3) should be more drastic [3]. This issue is not directly relevant for the present paper, which mainly addresses (1.2).

[2] Formulae with torsion higher than one were constructed in [15 – 17].

usually an arithmetic subgroup of $\mathrm{SL}(2,\mathbb{Z})$. For the STU model this is either the S-, the T- or the U-duality group, which equals the $\Gamma(2)$ subgroup of $\mathrm{SL}(2,\mathbb{Z})$.

The inverse of the automorphic form Φ_k takes the form of an infinite Fourier sum with certain powers of $\exp[\mathrm{i}\pi\rho]$, $\exp[\mathrm{i}\pi\sigma]$ and $\exp[\mathrm{i}\pi\upsilon]$, and the 3-cycle is then defined by choosing integration contours where the real parts of ρ, σ and υ take the appropriate values to select the Fourier modes in $1/\Phi_k$. Obviously the values taken by (K,L,M) must be correlated with the possible Fourier modes. The leading behaviour of the dyonic degeneracy is associated with the rational quadratic divisor $\mathcal{D} = \upsilon + \rho\sigma - \upsilon^2 = 0$ of Φ_k, near which $1/\Phi_k$ takes the form,

$$\frac{1}{\Phi_k(\rho,\sigma,\upsilon)} \approx \frac{1}{4\,\pi^2}\,\frac{1}{\mathcal{D}^2}\,\frac{1}{\Delta_k(\rho,\sigma,\upsilon)} + \mathcal{O}(\mathcal{D}^0)\,, \qquad \Delta_k = \frac{f^{(k)}(\gamma')\,f^{(k)}(\sigma')}{\sigma^{k+2}}\,, \tag{1.7}$$

where

$$\gamma' = \frac{\rho\sigma - \upsilon^2}{\sigma}\,, \qquad \sigma' = \frac{\rho\sigma - (\upsilon-1)^2}{\sigma}\,. \tag{1.8}$$

Here $f^{(k)}$ is a known modular form associated with $\mathrm{SL}(2,\mathbb{Z})$ or its appropriate subgroup. For the STU model we have $f^{(0)}(\gamma') = \vartheta_2^4(\gamma')$.

The choice of the divisor $\mathcal{D}$ strongly restricts possible redefinitions of the complex variables ρ,σ,υ. Both the exponential factor in (1.4) and the divisor are invariant under the dualities corresponding to (1.6), which implies,

$$\begin{aligned}
\rho &\to a^2\,\rho + b^2\,\sigma - 2\,ab\,\upsilon + ab\,,\\
\sigma &\to c^2\rho + d^2\,\sigma - 2\,cd\,\upsilon + cd\,,\\
\upsilon &\to -\,ac\,\rho - bd\,\sigma + (ad+bc)\upsilon - bc\,.
\end{aligned} \tag{1.9}$$

These transformations belong to the modular group $\mathrm{Sp}(2,\mathbb{Z})$ associated with Φ_k. The inhomogeneous terms in (1.9) contribute only to the real part of ρ,σ,υ, and they have some bearing on the periodicity intervals for the real values of ρ,σ,υ.

For the $\Gamma(2)$ subgroup of $\mathrm{SL}(2,\mathbb{Z})$, which is relevant for the STU model, we have $a,d = 1+2\,\mathbb{Z}$ and $b,c = 2\,\mathbb{Z}$, so that the real shifts induced in ρ,σ,υ are multiples of 2. This is consistent with the fact that $1/\Phi_0$ has a Fourier decomposition in terms of powers of $\exp[\mathrm{i}\pi\rho]$, $\exp[\mathrm{i}\pi\sigma]$ and $\exp[2\mathrm{i}\pi\upsilon]$, which implies that K, L and M must take integer values in order to find non-zero values for (1.4). Hence, the 3-cycle can be parametrized by,

$$0 \le \mathrm{Re}\,\sigma < 2\,, \qquad 0 \le \mathrm{Re}\,\rho < 2\,, \qquad 0 \le \mathrm{Re}\,\upsilon < 1\,. \tag{1.10}$$

The lattice of the charges p^I and q_I will be discussed in the next section.

The proposal of [10] for the dyon degeneracy in the STU model involves three integrals of the type (1.7), and reads,

$$d_{\mathrm{STU}}(p,q) = I_0(K_s,L_s,M_s)\;I_0(K_t,L_t,M_t)\;I_0(K_u,L_u,M_u)\,, \tag{1.11}$$

which is manifestly invariant under triality (related to interchanging the s, t and u labels), where the triplets of charge bilinears, (K_s,L_s,M_s), (K_t,L_t,M_t) and (K_u,L_u,M_u), transform as vectors under S-, T- and U-duality, respectively.

An asymptotic evaluation of the integral (1.4) can be done in the limit where $KL - M^2 \gg 1$ and $K + L$ is large and negative. Furthermore one assumes that $|K|$ is sufficiently small as compared to $\sqrt{KL - M^2}$. In this way one can recover non-perturbative string corrections, as was stressed in [18]. The evaluation of the integral (1.4) proceeds by first evaluating the contour integral for v around either one of the zeros $v_\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4\rho\,\sigma}$ of Φ_k on the divisor $\mathcal{D} = 0$. Subsequently, the two remaining integrals over ρ and σ are evaluated in saddle-point approximation. The saddle-point values of ρ, σ, and hence of $v_\pm$ are expressed in terms of σ' and γ' in a way that is independent of the choice of the pole position $v_\pm$. As it turns out, σ' and γ' can be identified with the complex modulus S in a field-theoretic description, according to $\gamma' = \mathrm{i}S$ and $\sigma' = \mathrm{i}\bar{S}$ [18]. In that case the saddle-point values of ρ, σ and v can be parametrized by

$$\rho = \frac{\mathrm{i}|S|^2}{S + \bar{S}} \,, \qquad \sigma = \frac{\mathrm{i}}{S + \bar{S}} \,, \qquad v = \frac{S}{S + \bar{S}} \,. \tag{1.12}$$

These values describe the unique solution to the saddle-point equations for which $d_k(K, L, M)$ takes a real value. The resulting expression for $\ln d_k(K, L, M)$ equals

$$\ln d_k(K, L, M) = \pi \left[-\frac{L - \mathrm{i}M(S - \bar{S}) + K|S|^2}{S + \bar{S}} - \frac{1}{\pi} \ln \Delta_k(S, \bar{S}) \right] \,, \tag{1.13}$$

where the right-hand side is evaluated at a stationary point, so that S (and therefore ρ, σ, v) will be determined in terms of K, L, M. In the limits specified earlier it turns out that S takes a finite value. The result then coincides precisely with the results obtained in the field-theoretic description [18, 13, 4], and it holds up to an additive constant and up to terms that are suppressed by inverse powers of the charges. Substituting the value for S and working to first order in Δ_k gives

$$\ln d_k(K, L, M) \approx \pi \sqrt{K\,L - M^2} - \ln \Delta_k(S, \bar{S}) \,. \tag{1.14}$$

This paper is organized as follows. In section 2 we consider the typical calculation of a mixed partition function, which will be used in the evaluation of the full mixed partition function for the STU model in section 3. In section 4 we present our conclusions.

2. Prototype evaluation of the mixed partition function

In the following, we compute the mixed partition function associated with $d_k(p, q)$ expressed by the integral (1.4). For definiteness we will be more specific here and consider the STU model, where $k = 0$. We follow the same strategy as in [6, 4] where the $N = 4$ supersymmetric models were considered.

As indicated in (1.11), the degeneracies for the STU model factorize into three integrals of the type (1.4), which are related by triality. Here we will first evaluate the mixed partition function as if there is only one such integral,

$$Z_s(p, \phi) = \sum_q d_0(K_s, L_s, M_s)\, \mathrm{e}^{\frac{1}{3}\pi\, q_I \phi^I} \,. \tag{2.1}$$

The reason for the factor $\frac{1}{3}$ in the exponent will become clear in the next section, where we will use the result of this calculation to obtain the full expression for the STU model based on (1.11).

According to [10], the fact that we have three such integrals implies that the quantities (K_s, L_s, M_s) must be equal to one-third of the charge bilinears $(\langle P,P\rangle_s, \langle Q,Q\rangle_s, \langle P,Q\rangle_s)$ that were used in the supergravity formulation (we use the notation of [3]). Hence we have (note that $I = 0,1,2,3$),

$$\begin{aligned} 3\,K_s &= \langle P,P\rangle_s = -2(p^0 q_1 + p^2 p^3)\ , \\ 3\,L_s &= \langle Q,Q\rangle_s = 2(q_0 p^1 - q_2 q_3)\ , \\ 3\,M_s &= \langle P,Q\rangle_s = q_0 p^0 - q_1 p^1 + q_2 p^2 + q_3 p^3\ . \end{aligned} \tag{2.2}$$

It is clear that the charges p^I and q_I cannot be integer-valued in this case, in view of the fact that the three quantities K_s, L_s and M_s must cover the same set of integer values. Combining various arguments presented in the previous section, we therefore conclude that the charges p^I and q_I take the following values,

$$p^{1,2,3}, q_0 \in \lambda^{-1}\,\mathbb{Z}\,, \qquad p^0, q_{1,2,3} \in \lambda\,\mathbb{Z}\,, \tag{2.3}$$

where $\lambda = \sqrt{2}$ or $\frac{1}{2}\sqrt{2}$, which is consistent with triality. The reader can easily verify that, based on the possible values of p^I and q_I, K_s, L_s and M_s cover the full range of integers (as well as the rational numbers $\mathbb{Z} \pm \frac{1}{3}$ for which I_0 will vanish). In the effective action the two values of λ are simply related by a uniform electric/magnetic duality transformation under which all p^I and q_I are interchanged. The above assignment is consistent with string theory where the STU model is described in terms of a freely acting $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of type-IIB string theory compactified on $T^4 \times S^1 \times \tilde{S}^1$. Here the choice of λ is related to the identification of the p^I and q_I with the momenta and winding numbers associated with the two circles, S^1 and $\tilde{S}^1$. We will not make a choice for λ in what follows in order to make the effect of the charge basis explicit in the calculation.

We now proceed and follow the derivation as presented in [4], recalling that while the charges p^I and q_I take the values given in (2.3), (1.4) will only be nonvanishing for $K_s, L_s, M_s \in \mathbb{Z}$. The integration contours are chosen according to (1.10), and $1/\Phi_0$ can be expanded in terms of Fourier coefficients $\exp \mathrm{i}\pi[m\rho + n\sigma + 2p\upsilon]$ with integers m, n, p.

We begin by summing over q_0 and q_1, replacing the sums over q_0 and q_1 in (2.1) by sums over the charges L_s and K_s, related by the identities,

$$q_0 = \frac{1}{2\,p^1}\,(3L_s + 2\,q_2 q_3)\ , \qquad q_1 = -\frac{1}{2\,p^0}\,\left(3K_s + 2\,p^2 p^3\right)\ . \tag{2.4}$$

We will be assuming that both p^0 and p^1 are non-vanishing and positive (the latter is only a matter of convenience). In doing so, we need to ensure that, when performing the sums over L_s and K_s, we only keep those contributions that lead to integer-valued charges of λq_0 and q_1/λ. This projection onto integer values can be implemented by inserting the series $N^{-1}\sum_{l=0}^{N-1} \exp[2\pi \mathrm{i}\, l\, P/N]$, where P and N are integers,[3] which projects onto all integer

[3]We assume that $N \geq 1$. Note that this formula remains correct when P and N have a common divisor.

values for P/N. The use of this formula leads to the following expression,

$$Z_s(p,\phi) = \frac{1}{4p^0p^1} \sum_{\substack{\phi^0\to\phi^0+6\mathrm{i}\,l^0\lambda \\ \phi^1\to\phi^1+6\mathrm{i}\,l^1/\lambda}} \sum_{L_s,K_s,q_2,q_3} d_0(L_s,K_s,M_s) \\ \times \exp\left[\frac{\pi\phi^0}{6p^1}(3L_s+2q_2q_3) - \frac{\pi\phi^1}{6p^0}(3K_s+2p^2p^3) + \frac{\pi}{3}\,(q_2\phi^2+q_3\phi^3)\right] , \tag{2.5}$$

with M_s given by

$$M_s = \frac{p^0}{6\,p^1}(3L_s+2\,q_2q_3) + \frac{p^1}{6\,p^0}(3K_s+2p^2p^3) + \frac{1}{3}(q_2p^2+q_3p^3)\,. \tag{2.6}$$

In (2.5) the summation over imaginary shifts of ϕ^0 and ϕ^1 is implemented by first replacing $\phi^0 \to \phi^0 + 6\,\mathrm{i}l^0\lambda$ and $\phi^1 \to \phi^1 + 6\,\mathrm{i}l^1/\lambda$ in each summand, and subsequently summing over the integers $l^0 = 0,\ldots,2p^1\lambda^{-1}-1$ and $l^1 = 0,\ldots,2p^0\lambda-1$. The sums over $l^{0,1}$ enforce that only those summands for which $(3\,L_s+2\,q_2q_3)\lambda/2p^1$ and $(3\,K_s+2\,p^2p^3)/2p^0\lambda$ are integers, give a non-vanishing contribution to $Z_s(p,\phi)$.

Next, consider summing over L_s without any restriction. Expanding $1/\Phi_k$ in Fourier modes,

$$\frac{1}{\Phi_0(\rho,\sigma,\upsilon)} = \sum_n \mathrm{e}^{\mathrm{i}\pi\,n\sigma} C_n(\rho,\upsilon)\ , \tag{2.7}$$

results in the double sum

$$\sum_{L_s,n} \mathrm{e}^{\mathrm{i}\pi\,[L_s(\sigma-\sigma(\upsilon))+n\,\sigma]}\, C_n(\rho,\upsilon)\ , \tag{2.8}$$

where we introduced

$$\sigma(\upsilon) = -\frac{\phi^0}{2\mathrm{i}p^1} - (2\upsilon-1)\frac{p^0}{2p^1}\,. \tag{2.9}$$

Subsequently, consider performing the contour integral of (2.8) over σ. This selects the Fourier mode $n=-L_s$, so that we obtain,

$$2\sum_{L_s} \mathrm{e}^{\mathrm{i}\pi L_s\sigma(\upsilon)}\, C_{L_s}(\rho,\upsilon) = \frac{2}{\Phi_0(\rho,\sigma(\upsilon),\upsilon)}\ . \tag{2.10}$$

Next, summing over K_s without any restriction and using analogous steps as described above, yields

$$\frac{4}{\Phi_0(\rho(\upsilon),\sigma(\upsilon),\upsilon)}\ , \tag{2.11}$$

where

$$\rho(\upsilon) = \frac{\phi^1}{2\mathrm{i}p^0} - (2\upsilon-1)\frac{p^1}{2p^0}\,. \tag{2.12}$$

Hence, after summing over L_s and K_s and performing two of the three contour integrals, we obtain

$$Z_s(p,\phi) = \frac{1}{p^0 p^1} \sum_{\substack{\phi^0\to\phi^0+6\mathrm{i}\,l^0\lambda \\ \phi^1\to\phi^1+6\mathrm{i}\,l^1/\lambda}} \sum_{q_2,q_3} \oint \mathrm{d}\upsilon\, \frac{1}{\Phi_0(\rho(\upsilon),\sigma(\upsilon),\upsilon)} \times \exp\Bigg(-\frac{1}{3}\mathrm{i}\pi\Big[2\,\sigma(\upsilon)\,q_2q_3 + 2\,\rho(\upsilon)\,p^2p^3 + \mathrm{i}q_2(\phi^2+\mathrm{i}(2\upsilon-1)p^2) + \mathrm{i}q_3(\phi^3+\mathrm{i}(2\upsilon-1)p^3)\Big]\Bigg)\,. \quad (2.13)$$

The integrand is manifestly invariant under the shifts $\phi^0 \to \phi^0 + 3\,\mathrm{i}p^1$, $\phi^1 \to \phi^1 + 12\,\mathrm{i}p^0$ (or $\phi^0 \to \phi^0 + 12\,\mathrm{i}p^1$, $\phi^1 \to \phi^1 + 3\,\mathrm{i}p^0$, depending on the value of λ) and $\phi^{2,3} \to \phi^{2,3} + 6\mathrm{i}\lambda^{-1}$, so that the explicit sum over shifts with $l^0 = 0,\ldots,2p^1\lambda^{-1}-1$ and $l^1 = 0,\ldots,2p^0\lambda-1$ ensures that the partition function (2.13) is invariant under any shifts of $\phi^{1,2,3}$ that are multiples of $6\mathrm{i}\lambda^{-1}$ and of ϕ^0 that are multiples of $6\mathrm{i}\lambda$. Note that we are overcounting in this way, because the full range of the explicit sum over shifts of either ϕ^0 or ϕ^1 is not required in view of the explicit invariance of the integrand. In this particular case this will lead to an irrelevant multiplicative factor 4. In practice we will impose an infinite sum over shifts for all the fields ϕ, while modding out the shifts that correspond already to an invariance in the final result. In this way we respect the symmetry of the initial expression (1.1).

Subsequently we perform a formal Poisson resummation over the charges q_2 and q_3, and obtain[4]

$$Z_s(p,\phi) = -\frac{3\,\mathrm{i}}{\lambda^2 p^0 p^1} \sum_{\text{shifts}} \oint \mathrm{d}\upsilon\, \frac{1}{\sigma(\upsilon)\;\Phi_0(\rho(\upsilon),\sigma(\upsilon),\upsilon)} \times \exp\left(-\frac{1}{3}\mathrm{i}\,\pi\left[2\,p^2p^3\,\rho(\upsilon) + \frac{(\phi^2+\mathrm{i}(2\upsilon-1)p^2)\,(\phi^3+\mathrm{i}(2\upsilon-1)p^3)}{2\sigma(\upsilon)}\right]\right)\,, \quad (2.14)$$

where the sum over shifts now also includes an infinite sum over multiple the shifts $\phi^{2,3} \to \phi^{2,3} + 6\mathrm{i}\lambda^{-1}$, which are induced by the Poisson summation. Note that the invariance over the shifts $\phi^0 \to \phi^0 + 3\,\mathrm{i}p^1$ (or $\phi^0 \to \phi^0 + 12\,\mathrm{i}p^1$) is no longer manifest after the resummation.

Now we perform the contour integral over υ. This integration picks up the contributions from the residues at the various poles of the integrand. We assume that the leading contribution to this sum of residues stems from the rational quadratic divisor $\mathcal{D} = \upsilon + \rho\sigma - \upsilon^2 = 0$ of Φ_0. Other poles of the integrand in (2.14) are expected to give rise to exponentially suppressed contributions in the limit that the charges are large. Inserting $\rho(\upsilon)$ and $\sigma(\upsilon)$ into $\mathcal{D}$ yields

$$\mathcal{D} = 2(\upsilon-\upsilon_*)\,\frac{\phi^0p^1-\phi^1p^0}{4\mathrm{i}p^0p^1}\,, \quad (2.15)$$

with υ_* given by

$$2\upsilon_* = 1 - \mathrm{i}\,\frac{\phi^0\phi^1+p^1p^0}{\phi^0p^1-\phi^1p^0}\,. \quad (2.16)$$

[4]The resummation involves a (divergent) gaussian integral, which can be evaluated upon performing an analytic continuation of the integration variables. We assume that this continuation leads to an overall factor $-\mathrm{i}$. Observe that we are interested in the result for imaginary values of $\sigma(\upsilon)$, as is explained below.

The corresponding values of $\rho_* = \rho(\upsilon_*)$ and $\sigma_* = \sigma(\upsilon_*)$ take the following form,

$$\sigma_* = \frac{\mathrm{i}}{2}\,\frac{(\phi^0)^2+(p^0)^2}{\phi^0 p^1 - \phi^1 p^0}\,, \qquad \rho_* = \frac{\mathrm{i}}{2}\,\frac{(\phi^1)^2+(p^1)^2}{\phi^0 p^1 - \phi^1 p^0}\,. \tag{2.17}$$

We observe that $\mathcal{D}$ has only a simple zero. Using (1.7) we can perform the contour integral over υ, which yields,

$$\begin{aligned} Z_s(p,\phi) = &-\frac{6\,p^0p^1}{\pi\lambda^2}\sum_{\text{shifts}}\frac{1}{(\phi^0p^1-\phi^1p^0)^2} \\ &\times\frac{\mathrm{d}}{\mathrm{d}\upsilon}\left[\frac{\exp\left(-\frac{1}{3}\mathrm{i}\pi\left[2p^2p^3\,\rho(\upsilon)+\frac{(\phi^2+\mathrm{i}(2\upsilon-1)p^2)\,(\phi^3+\mathrm{i}(2\upsilon-1)p^3)}{2\sigma(\upsilon)}\right]\right)}{\sigma(\upsilon)\,\Delta_0(\rho(\upsilon),\sigma(\upsilon),\upsilon)}\right]_{\upsilon=\upsilon_*}. \end{aligned} \tag{2.18}$$

Evaluating this expression leads to (we refer to [4] for additional details),

$$\begin{aligned} Z_s(p,\phi) = \frac{2}{\lambda^2}\sum_{\text{shifts}}&\left[\frac{\pi(p^2\phi^0-p^0\phi^2)(p^3\phi^0-p^0\phi^3)-6\mathrm{i}p^0\sigma_*\left(p^0+p^1\sigma_*\frac{\mathrm{d}\ln[\sigma^2\Delta_0]}{\mathrm{d}\upsilon}\Big|_*\right)}{\pi\,((\phi^0)^2+(p^0)^2)(\phi^0p^1-\phi^1p^0)}\right] \\ &\times\exp\left[\frac{1}{3}\pi\mathcal{F}_0(p,\phi)-\ln[\sigma_*^2\,\Delta_0(\rho_*,\sigma_*,\upsilon_*)]\right] \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} \mathcal{F}_0(p,\phi) = \frac{-1}{(\phi^0)^2+(p^0)^2}\Big[&\phi^0(p^1\phi^2\phi^3+p^2\phi^3\phi^1+p^3\phi^1\phi^2) \\ &+p^0(\phi^1p^2p^3+\phi^2p^3p^1+\phi^3p^1p^2)-p^0\phi^1\phi^2\phi^3-\phi^0p^1p^2p^3\Big]\,, \end{aligned} \tag{2.20}$$

which is manifestly invariant under triality.

We close this section by indicating the relationship with various quantities that appear in the macroscopic description of the STU model. First we define Y^I by

$$Y^I = \frac{1}{2}(\phi^I + \mathrm{i}p^I)\,, \tag{2.21}$$

and we introduce the ratios $\mathrm{i}S = Y^1/Y^0$, $\mathrm{i}T = Y^2/Y^0$ and $\mathrm{i}U = Y^3/Y^0$. It then follows straightforwardly that

$$\rho_* = \frac{\mathrm{i}|S|^2}{S+\bar{S}}\,, \quad \sigma_* = \frac{\mathrm{i}}{S+\bar{S}}\,, \quad \upsilon_* = \frac{S}{S+\bar{S}}\,, \tag{2.22}$$

which coincides with (1.12). In this parametrization it is easy to show that $\mathrm{Im}(\rho_*)\,\mathrm{Im}(\sigma_*) - (\mathrm{Im}(\upsilon_*))^2 = \frac{1}{4}$, so that the point $(\rho_*,\sigma_*,\upsilon_*)$ is located on the Siegel upper-half plane. Furthermore, we find that

$$\omega(p^0,p^1,\phi^0,\phi^1) \equiv \sigma_*^2\,\Delta_0(\rho_*,\sigma_*,\upsilon_*) = f^{(0)}(\mathrm{i}S)\,f^{(0)}(\mathrm{i}\bar{S})\,. \tag{2.23}$$

JHEP12(2008)086

Subsequently we consider the function

$$F(Y) = -\frac{Y^1 Y^2 Y^3}{Y^0}\,, \tag{2.24}$$

and establish that

$$\mathcal{F}_0(p,\phi) = 4\,\mathrm{Im}[F(Y)]\,. \tag{2.25}$$

The mixed free energy $\mathcal{F}_{\rm E}(p,\phi)$ of the STU model equals,

$$\mathcal{F}_{\rm E}(p,\phi) = \mathcal{F}_0(p,\phi) - \frac{1}{\pi}\left[\ln\omega(p^0,p^1,\phi^0,\phi^1) + \ln\omega(p^0,p^2,\phi^0,\phi^2) + \ln\omega(p^0,p^3,\phi^0,\phi^3)\right]\,. \tag{2.26}$$

Now consider the limit where the charges p^I and the ϕ^I are large. The leading part in the prefactor in (2.19) then equals

$$\mathrm{e}^{-\mu_s(p,\phi)} \equiv \frac{(p^2\phi^0 - p^0\phi^2)(p^3\phi^0 - p^0\phi^3)}{((\phi^0)^2 + (p^0)^2)(\phi^0 p^1 - \phi^1 p^0)} = \frac{(T+\bar{T})(U+\bar{U})}{2\,(S+\bar{S})}\,, \tag{2.27}$$

where we note the expression for the Kähler potential $\mathcal{K}$

$$\mathcal{K} = -\ln[(S+\bar{S})(T+\bar{T})(U+\bar{U})] = -\ln\left[\frac{\mathrm{i}(\bar{Y}^I F_I - Y^I \bar{F}_I)}{|Y^0|^2}\right]\,, \tag{2.28}$$

where $F_I = \partial F/\partial Y^I$.

3. The mixed partition function for the STU model

In this section, we evaluate the full mixed partition function $Z_{\rm STU}(p,\phi)$ for the STU model. In order to make use of the results obtained in the previous section, we write $Z_{\rm STU}(p,\phi)$ as

$$\begin{aligned} Z_{\rm STU}(p,\phi) &= \sum_{\{q\}} d_{\rm STU}(q,p)\,\mathrm{e}^{\pi q_I\,\phi^I} \\ &= \sum_{\{q,q',q''\}} \delta_{q,q'}\,\delta_{q',q''}\,d_0(K_s,L_s,M_s)\,d_0(K_t,L_t,M_t)\,d_0(K_u,L_u,M_u) \\ &\qquad \times \mathrm{e}^{\frac{\pi}{3}\left[\left(q_0+q_0'+q_0''\right)\phi^0 + \left(q_1+q_1'+q_1''\right)\phi^1 + \left(q_2+q_2'+q_2''\right)\phi^2 + \left(q_3+q_3'+q_3''\right)\phi^3\right]}\,, \end{aligned} \tag{3.1}$$

with (K_s,L_s,M_s) given by (2.2), and where (K_t,L_t,M_t) and (K_u,L_u,M_u) follow by triality, except that at the same time we change the charges q to q' and q'', respectively,

$$\begin{aligned} 3\,K_t &= -2(p^0 q_2' + p^1 p^3)\,, \\ 3\,L_t &= 2(q_0' p^2 - q_1' q_3')\,, \\ 3\,M_t &= q_0' p^0 - q_2' p^2 + q_1' p^1 + q_3' p^3\,, \\ 3\,K_u &= -2(p^0 q_3'' + p^1 p^2)\,, \\ 3\,L_u &= 2(q_0'' p^3 - q_1'' q_2'')\,, \\ 3\,M_u &= q_0'' p^0 - q_3'' p^3 + q_2'' p^2 + q_1'' p^1\,. \end{aligned} \tag{3.2}$$

We will be assuming that all charges p^I are nonzero and positive.

The insertion of the Kronecker deltas leads to three copies (one for each of the three sectors S, T and U) of the mixed partition function computed in section 2. These copies, $Z_s(p,\phi_s)$, $Z_t(p,\phi_t)$ and $Z_u(p,\phi_u)$, are related to the each other by triality. Using the representation for the delta symbol (with integers n,m),

$$\delta_{mn} = \int_0^1 \mathrm{d}\theta\ \mathrm{e}^{2\mathrm{i}\pi(m-n)\theta}\,, \tag{3.3}$$

we rewrite (3.1) as follows,

$$Z_{\mathrm{STU}}(p,\phi) = \int_0^1 \mathrm{d}^4\theta\,\mathrm{d}^4\varphi\ Z_s(p,\phi_s)\,Z_t(p,\phi_t)\,Z_u(p,\phi_u)\,, \tag{3.4}$$

where

$$\begin{aligned}
\phi_s{}^0 &= \phi^0 + 6\mathrm{i}\lambda\,\theta^0\,, & \phi_s{}^{1,2,3} &= \phi^{1,2,3} + 6\mathrm{i}\lambda^{-1}\,\theta^{1,2,3}\,,\\
\phi_t{}^0 &= \phi^0 + 6\mathrm{i}\lambda\,(\varphi^0 - \theta^0)\,, & \phi_t{}^{1,2,3} &= \phi^{1,2,3} + 6\mathrm{i}\lambda^{-1}\,(\varphi^{1,2,3} - \theta^{1,2,3})\,,\\
\phi_u{}^0 &= \phi^0 - 6\mathrm{i}\lambda\,\varphi^0\,, & \phi_u{}^{1,2,3} &= \phi^{1,2,3} - 6\mathrm{i}\lambda^{-1}\,\varphi^{1,2,3}\,.
\end{aligned} \tag{3.5}$$

Observe that

$$\phi_s{}^I + \phi_t{}^I + \phi_u{}^I = 3\,\phi^I\,. \tag{3.6}$$

We remind the reader that each of the factors Z_s, Z_t and Z_u is invariant under the shifts $\phi^{1,2,3} \to \phi^{1,2,3} + 6\mathrm{i}\lambda^{-1}$ and $\phi^0 \to \phi^0 + 6\mathrm{i}\lambda$, by virtue of the (finite or infinite) explicit sums contained in these factors. Note that the infinite shift sums occur for $\phi_s{}^{2,3}$, $\phi_t{}^{1,3}$, and $\phi_u{}^{1,2}$, while the remaining shift sums cover a finite range. As it turns out most of these sums can be generated by extending the integrals over θ^I and φ^I from the interval $[0,1]$ to a larger interval. To explain this, consider the integration over φ^2 and θ^2. The factors Z_s and Z_u contain both an infinite sum of shifts of ϕ^2, whereas Z_t contains a finite sum of such shifts. The two infinite sums are thus included by extending the range of integration of φ^2 and θ^2 from $[0,1]$ to $[-\infty,\infty]$. In this way we are left with one finite sum over shifts of ϕ^2 (whose range is determined by the value of p^0) and two integrals ranging over $\varphi^2,\theta^2 \in [-\infty,\infty]$. The same procedure applies to the integration over $\varphi^{1,3}$ and $\theta^{1,3}$. Concerning the integration over φ^0 and θ^0, the situation is slightly different, because each of the three factors Z_s, Z_t and Z_u involves a finite sum of shifts of ϕ^0, and each is invariant under $\phi^0 \to \phi^0 + 12\mathrm{i}p^a$ with $a = 1,2,3$, respectively. This implies that the first of the three finite sums can be used to extend the range of integration of θ^0 from $[0,1]$ to $[0, 2p^1\lambda^{-1}]$, the second sum can be used to extend the range of integration of φ^0 to $[0, 2p^2\lambda^{-1}]$, while the third sum is kept untouched. Here we may be overcounting slighly as we explained in the text below (2.13), depending on the choice for λ, but this does not present a problem of principle. Below we will evaluate the resulting expression for large charges and large potentials, in which case one extends all the ranges of integration to the infinite interval $[-\infty,\infty]$ and sums over all the shifts $\phi^{1,2,3} \to \phi^{1,2,3} + 6\mathrm{i}\lambda^{-1}$ and $\phi^0 \to \phi^0 + 6\mathrm{i}\lambda$ at the end. Hence we consider the following integral,

$$Z_{\mathrm{STU}}(p,\phi) = \sum_{\phi^I-\text{shifts}} \int_{-\infty}^{\infty} \mathrm{d}^4\theta\,\mathrm{d}^4\varphi\ Z_s(p,\phi_s)\,Z_t(p,\phi_t)\,Z_u(p,\phi_u)\,, \tag{3.7}$$

where Z_s, Z_t and Z_u follow from (2.19), but without the explicit sum over the imaginary shifts, which have now been incorporated in the sum over the ϕ^I-shifts and in the extended φ^I- and θ^I-integration domains. We will evaluate this integral in saddle-point approximation. Before doing so we consider the integrand in somewhat more detail,

$$\begin{aligned} Z_s(p,\phi_s)\, Z_t(p,\phi_t)\, Z_u(p,\phi_u) \approx \qquad\qquad (3.8)\\ = \exp\Big\{\frac{1}{3}\pi[\mathcal{F}_0(p,\phi_s)+\mathcal{F}_0(p,\phi_t)+\mathcal{F}_0(p,\phi_u)]\\ -[\ln\omega(p^0,p^1,\phi_s^0,\phi_s^1)+\ln\omega(p^0,p^2,\phi_t^0,\phi_t^2)+\ln\omega(p^0,p^3,\phi_u^0,\phi_u^3)]\\ -[\mu_s(p,\phi_s)+\mu_t(p,\phi_t)+\mu_u(p,\phi_u)]\Big\}, \end{aligned}$$

where the $\mathcal{F}_0$ was defined in (2.20), whereas the expressions for ω and μ follow from the ones given in (2.23) and (2.27) by triality. Here we suppressed the terms in (2.19) that vanish in the limit of large charges p^I and large potentials ϕ^I. In that same limit, the contributions contained in ω and μ are subleading relative to those contained in $\mathcal{F}_0$ and can therefore be ignored when evaluating (3.7) in saddle-point approximation. We therefore expand $\mathcal{F}_0(p,\phi_s)+\mathcal{F}_0(p,\phi_t)+\mathcal{F}_0(p,\phi_u)$ in powers of θ^I and φ^I. The terms linear in θ^I and φ^I all cancel out by virtue of (3.6), so that we have a saddle point at $\theta^I=\varphi^I=0$. The term quadratic in θ^I and φ^I is homogeneous of zeroth degree in (p^I,ϕ^I), whereas higher powers in θ^I and φ^I have coefficients that are homogeneous of negative degree. This indicates that possible other saddle points will be exponentially suppressed. Retaining only the terms quadratic in θ^I and φ^I one may perform the corresponding eight-dimensional gaussian integral, which turns out to be equal (possibly up to a multiplicative constant) to $\exp[2\,\mathcal{K}]$, where $\mathcal{K}$ is given by (2.28). The calculation leading to this result is rather non-trivial. An easier exercise is to derive this result in the special case of $p^0=0$.

Combining this result with the terms independent of θ^I and φ^I thus leads to the result,

$$Z_{\rm STU}(p,\phi) \approx \sum_{\phi-\text{shifts}} \mathrm{e}^{\pi\,\mathcal{F}_{\rm E}(p,\phi)+\mathcal{K}}\,, \qquad (3.9)$$

up to an overall numerical constant. Here the mixed free energy, $\mathcal{F}_{\rm E}(p,\phi)$, was defined in (2.26), and we used that $\exp[-\mu_s(p,\phi)-\mu_t(p,\phi)-\mu_u(p,\phi)]=\frac{1}{8}\exp[-\mathcal{K}]$. The multiplicative factor $\exp[\mathcal{K}]$ is in precise agreement with the one conjectured in [3] on the basis of semiclassical arguments for a class of $N=2$ theories which includes the STU model.

4. Discussion and conclusions

The result obtained in the previous section demonstrates that the proposal of [10] for the dyonic degeneracies of the STU model leads to the mixed partition function with a prefactor that agrees with the prediction of [3]. The result was obtained in the case that all charges p^I are non-zero and positive, in the limit of large charges and large potentials ϕ^I. The charges were only taken positive to simplify the formulae, and we expect that there exists a similar result for $p^0=0$. In the latter case, an alternative, but rather similar, calculation

JHEP12(2008)086

seems possible provided that $p^1, p^2, p^3 \neq 0$. Based on previous experience [6, 4], we expect an analogous result.

The agreement that we have established here lends further support to the approaches taken in [10] and [3], and goes beyond the fact that the leading and subleading contributions to the entropy are in agreement (up to certain subtleties that we will again discuss below). The two approaches are based on entirely different considerations. Unlike in $N = 4$ models, we were forced to rely on a saddle-point approximation of the integral (3.7) at the end of the calculation, but the major part of the calculation does not depend on that. Therefore the result could a priori have been different. In fact there are other predictions in the literature [24] for the prefactor in (3.9), derived in a different regime. For a variety of reasons it seems unlikely that the present calculation can shed some light on these different results. Some of these reasons are discussed below.

As was stressed in [3], there is a distinct difference between the dyonic degeneracies for the various $N = 4$ models proposed earlier and the expression for the dyonic degeneracies in the STU model, which was already exhibited in [10]. The remarkable feature of the $N = 4$ models is that the saddle-point equations for the leading and subleading terms (c.f. (1.13)), which determine the entropy of large black holes from the microscopic degeneracies, coincide with the attractor equations of supergravity [18, 19]. This feature might be due to the high degree of symmetry in $N = 4$ models. For the STU model this relationship does not hold, although the statistical and the macroscopic entropy still agree to this order. Though this difference in behaviour of the dyonic degeneracy formula for the STU model from that of the $N = 4$ models does not, perhaps, indicate any fundamental inconsistency, it warrants at least a closer study of the next subleading correction to the entropy.

Another remarkable feature of the dyonic degeneracy formula for $N = 4$ models is that its form remains the same across walls of marginal stability. The dependence of the degeneracies on the asymptotic moduli is encoded in the choice of the integration contour used for extracting the degeneracies from (1.4) [20 – 22]. When the asymptotic moduli cross walls of marginal stability, the dyon can decay into a pair of 1/2-BPS states. Let us focus on a wall of marginal stability at which a 1/4-BPS dyon decays into a pair of purely electric and purely magnetic 1/2-BPS states,

$$(Q, P) \to (Q, 0) + (0, P)\,. \tag{4.1}$$

Then, by general arguments [23, 24], the degeneracy of 1/4-BPS states jumps across such a wall and the change is given by

$$d_>(Q, P) - d_<(Q, P) = (Q \cdot P)\,(-1)^{(Q\cdot P)+1}\, d_{\rm el}(Q)\, d_{\rm mag}(P)\,, \tag{4.2}$$

where $d_>(Q, P)$ and $d_<(Q, P)$ refer to the degeneracies of 1/4-BPS states across the wall, and $d_{\rm el}(Q)$ and $d_{\rm mag}(P)$ refer to the degeneracy of the purely electric and purely magnetic 1/2-BPS states. This wall crossing formula is obeyed by the $N = 4$ dyon degeneracy formula, because the modular form factorizes across the divisor $\upsilon = 0$ as

$$\Phi_k(\rho, \sigma, \upsilon) \sim 4\pi^2\, \upsilon^2\, g^{(k)}_{\rm mag}(\rho)\, g^{(k)}_{\rm el}(\sigma)\,, \tag{4.3}$$

where $g^{(k)}_{\rm el}(\sigma)$ denotes the partition function for purely electric states and $g^{(k)}_{\rm mag}(\rho)$ denotes the partition function for purely magnetic states. Then the jump in (4.2) arises due to the contribution of the double pole at $\upsilon \sim 0$. This feature of the $N = 4$ degeneracy formulae ensures that the function retains the same form across a wall of marginal stability and that the degeneracies can just be extracted by an appropriate choice of the integration contour.

Let us examine whether the above feature is present in the partition function of dyons in the STU model. The STU model also admits a wall of marginal stability at which the dyon decays acoording to (4.1), and we may consider whether the corresponding automorphic form admits a similar factorization as in (4.3). Since the partition function is a product of three modular forms $\Phi_0(\rho, \sigma, \upsilon)$, there are three divisors, $\upsilon_s = 0$, $\upsilon_t = 0$ and $\upsilon_u = 0$. At, say, the divisor $\upsilon_s = 0$ and $\upsilon_t, \upsilon_u \neq 0$, the degeneracy formula factorizes as (see [10] for the properties of Φ_0)

$$\begin{aligned} &\Phi_0(\rho_s, \sigma_s, \upsilon_s)\ \Phi_0(\rho_t, \sigma_t, \upsilon_t)\ \Phi_0(\rho_u, \sigma_u, \upsilon_u) \\ &\quad \sim 4\pi^2\, {\upsilon_s}^2\, \frac{\eta^8(2\rho_s)}{\eta^4(\rho_s)}\, \frac{\eta^8(\sigma_s/2)}{\eta^4(\sigma_s)}\, \Phi_0(\rho_t, \sigma_t, \upsilon_t)\ \Phi_0(\rho_u, \sigma_u, \upsilon_u)\ . \end{aligned} \tag{4.4}$$

The contribution of this double pole to the degeneracy is of the form

$$M_s\, (-1)^{M_s+1}\, d_1(K_s)\ d_2(L_s)\ I_0(K_t, L_t, M_t)\ I_0(K_u, L_u, M_u)\ , \tag{4.5}$$

where

$$d_1(K_s) = \oint {\rm d}\rho\, \frac{{\rm e}^{{\rm i}\pi K_s \rho}}{\eta^8(2\rho)\, \eta^{-4}(\rho)}\ , \qquad d_2(L_s) = \oint {\rm d}\sigma \frac{{\rm e}^{{\rm i}\pi L_s \sigma}}{\eta^8(\sigma/2)\, \eta^{-4}(\sigma)}\ . \tag{4.6}$$

Certainly (4.5) does not obey the wall crossing formula (4.2). The same conclusion holds at the other divisors $\upsilon_t = 0$ or $\upsilon_u = 0$, or combinations thereof. This suggests that the degeneracy formula (1.11) is valid only in the region of asymptotic moduli where the singlecentered black hole is stable. Restricting the domain of validity of the partition function to such a region avoids the entropy enigma, because the multicentered solutions found by [24], which dominate the entropy, are not stable in that case. It will be interesting to study this region by carefully considering the walls of marginal stability for the STU model. Across walls of marginal stability, the partition function should be modified in such a way that the wall crossing formula holds. Therefore such a study can perhaps indicate how the degeneracy formula can be extended to other domains.

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JHEP12(2008)086